Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Simple Model for Particle Radiative Transfer in Vacuum Particle Plumes

D. L. Cohen*
Massachusetts Institute of Technology, Lexington, Massachusetts

Introduction

SOLID fuel rocket motors are routinely used above the Earth's atmosphere for staging and orbit changes. The fuel consumption rate of these motors can be as high as 2000 g/s, and their exhaust plumes often contain large numbers of micron-sized particles.[1] The exhaust plume particles are typically ejected at temperatures on the order of 2000 K or higher, and have velocities on the order of 5000 ft/s. Many times it is of interest to know how these exhaust particles cool off radiatively as they travel down the plume. To do this accurately, we must account not only for the particles radiating thermal energy to infinity but also for the interchange of radiant thermal energy between the particles as they travel. Although work has been done in this area before, it did not address the question of when the particles' self-interaction can be neglected and when it cannot.[2–4]

Analysis

Figure 1 shows the basic plume model that will be analyzed here. The particles are identical spheres of radius R traveling at a constant radial velocity v. They are at a great distance from their source, so the exact mechanism used to produce them is of no great significance. We assume that when the particles reach $r = r_{min}$ they all have the same temperature $T = T_b$. The particle concentration is zero for $\theta > \theta_{max}$ and (ψ/r^2) for $\theta \le \theta_{max}$, where ψ is a constant having units of inverse length. The equation describing a particle's thermal energy E as a function of time t is

$$\frac{dE}{dt} = -H_{particle} + H_{ext} + H_{plume} \tag{1}$$

where $H_{particle}$ = rate at which the particle radiates heat = $(4\pi R^2 \epsilon\sigma T^4)$, T = particle temperature, ϵ = gray particle surface emissivity, σ = Stefan-Boltzmann constant, H_{ext} = rate at which heat is absorbed from sources outside the plume, and H_{plume} = rate at which heat is absorbed from other particles in the plume. We assume in Eq. (1) that $H_{particle} \gg H_{ext}$ and $H_{particle} \gg H_{plume}$, but that H_{ext} and H_{plume} are not so small that they can be ignored safely. The size of H_{ext} will be determined by the type of external radiant heat sources existing near the plume (for example, the rocket motor and its associated exhaust at $r < r_{min}$).

The formula for H_{plume} is basically an integral over the volume of the plume for all values of $r \ge r_{min}$, $\theta \le \theta_{max}$.

$$H_{plume}(\mathbf{r}) = \int d^3r' \left(\frac{\Psi}{r'^2}\right)\left(\frac{\epsilon\pi R^2}{|\mathbf{r}' - \mathbf{r}|^2}\right) R^2 \epsilon\sigma T^4(\mathbf{r}'); \quad r \ge r_{min},\ \theta \le \theta_{max} \tag{2}$$

where $T(\mathbf{r}')$ = temperature of a particle at position $\mathbf{r}'$, ε = gray particle surface absorptivity and emissivity, $\mathbf{r}$ = position vector of the particle absorbing radiation emitted by the rest of the particle cloud, $d^3\mathbf{r}$ = differential volume $(= r^2 \sin\theta \, d\theta \, d\phi \, dr)$ and $r' = |\mathbf{r}'|$ = length of position vector from the origin of the spherical coordinate system defined in Fig. 1.

The particles in the plume will have a temperature that depends only on position, since the rocket plume is a time-independent system. Because the particles all travel at the same radial velocity v, for any one particle, $dt = dr/v$.

$$\frac{dE}{dt} = \frac{4}{3}\pi R^3 \rho C \frac{dT}{dt} = \frac{4}{3}\pi R^3 \rho C v \frac{\partial T}{\partial r} \tag{3}$$

where ρ = particle mass density and C = particle heat capacity per unit mass. It is helpful to create a new temperature scale $\tau = T/T_b$ and to create a new length scale $s = kr$, where $k = (9\varepsilon\sigma T_b^3)/(\rho v R C)$. We then combine Eqs. (1–3) to get

$$\frac{\partial\tau}{\partial s} = -\frac{\tau^4}{3} + \alpha h_e + \beta \int_{s_{min}}^{\infty} ds' \int_0^{\theta_{max}} d\theta' \sin\theta' \int_0^{2\pi} d\phi' \frac{\tau^4(s')}{|\mathbf{s}' - \mathbf{s}|^2}, \quad \tau(s_{min}) = 1 \tag{4}$$

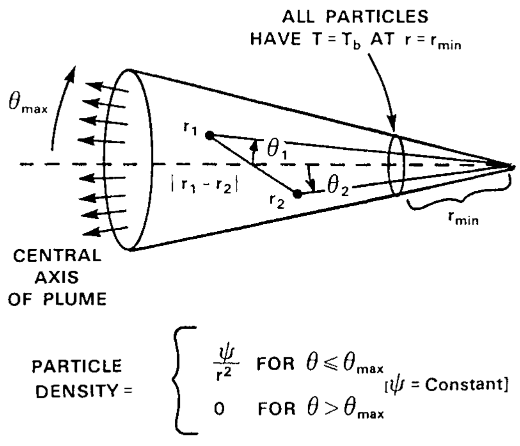

Fig. 1 Simplified model of the far-field particle exhaust plume.

Received May 27, 1987; revision received Aug. 10, 1987.

*Staff Member, Group 95, Countermeasures Technology, Lincoln Laboratory.

where H_e = typical rate at which heat is absorbed by the particle from radiating sources external to the plume, $\alpha = H_e/(12\pi R^2 \varepsilon \sigma T_b^4)$, $\beta = \varepsilon k R^2 \psi/12$, $h_e = H_{\text{ext}}/H_e = O(1)$ quantity, and $s_{\min} = kr_{\min}$. For the rocket motors described earlier, we take $R \sim 5\ \mu$, $\rho \sim 2\ \text{g/cm}^3$ (the density of bulk carbon), $C \sim 5 \times 10^6$ erg/g/K (the heat capacity of carbon), $\varepsilon \sim 1$, and $\theta_{\max} \sim 5$ deg. Evaluating k and β, we find that k is of order $10^{-3}\ \text{cm}^{-1}$, while β is of order 10^{-1}. Since τ^4 is $O(1)$, the integral term in Eq. (4)—although small—should not be neglected. From now on it will be assumed that both α and β are reasonably small parameters so that a perturbation expansion can be used to solve for $\tau(s)$.

$$\tau(s) = \tau_0(s) + \alpha\tau_A(s) + \beta\tau_B(s) + \text{higher order terms in } \alpha \text{ and } \beta \tag{5}$$

Equation (5) is substituted into Eq. (4) and $O(1)$ terms, the $O(\beta)$ terms, and the $O(\alpha)$ terms are collected into three sets of equations.

$$\frac{\partial\tau_0}{\partial s} = -\frac{\tau_0^4}{3} \quad \text{with } \tau_0(s_{\min}) = 1 \tag{6a}$$

$$\frac{\partial\tau_A}{\partial s} = -\frac{4}{3}\tau_0^3\tau_A + h_e \quad \text{with } \tau_A(s_{\min}) = 0 \tag{6b}$$

$$\frac{\partial\tau_B}{\partial s} = -\frac{4}{3}\tau_0^3\tau_B + \int_{s_{\min}}^{\infty} \mathrm{d}s'\tau_0^4(s') \int_0^{\theta_{\max}} \mathrm{d}\theta' \sin\theta' \int_0^{2\pi} \frac{\mathrm{d}\phi'}{|s'-s|^2} \quad \text{with } \tau_B(s_{\min}) = 0 \tag{6c}$$

Note how the boundary conditions specified in Eqs. (6) ensure that $\tau(s)$ in Eq. (5) will satisfy the boundary condition given in Eq. (4).

Clearly, τ_0 in Eq. (6a) will be a function only of $|s| = s$, and, in fact, we find that

$$\tau_0(s) = \frac{1}{(\gamma + s)^{\frac{1}{3}}} \tag{7}$$

where $\gamma = 1 - s_{\min}$.

It is hard to say much about τ_A in Eq. (6b) without knowing the exact form of function h_e; however, the formal solution can be written as

$$\tau_A(s,\theta,\phi) = \tau_0^4(s) \int_{1-\gamma}^{s} h_e(s',\theta,\phi)(\gamma + s')^{\frac{4}{3}}\,\mathrm{d}s' \tag{8}$$

In Eq. (6c), the volume integral plays the role of h_e so, therefore, we have

$$\tau_B(s'',\theta) = \tau_0^4(s'') \int_{1-\gamma}^{s''} \mathrm{d}s(\gamma + s)^{\frac{4}{3}} \int_{1-\gamma}^{\infty} \frac{\mathrm{d}s'}{(\gamma + s')^{\frac{4}{3}}} \times \int_0^{\theta_{\max}} \mathrm{d}\theta' \sin\theta' \int_0^{2\pi} \frac{\mathrm{d}\phi'}{|s'-s|^2} \tag{9}$$

In Eq. (9), it is assumed that $\theta_{\max} \ll 1$ so that $\cos(x) \approx 1 - x^2/2$ and $\sin(x) \approx x$ for $x = \theta, \theta'$. By consulting a standard handbook of indefinite integrals and integrating over ϕ' and θ', the formula for τ_B can be written as a double integral over the two dummy variables p and q.[5]

$$\tau_B(s,\theta) = \pi\tau_0^4(s)F(s,\theta,\theta_{\max},\gamma) \tag{10}$$

where

$$F(s,\theta,\theta_{\max},\gamma) = \int_{1-\gamma}^{s} \frac{\mathrm{d}p(\gamma + p)^{\frac{4}{3}}}{p} \int_{1-\gamma}^{\infty} \mathrm{d}q \frac{G(q,p,\theta,\theta_{\max})}{q(\gamma + q)^{\frac{4}{3}}} \tag{11}$$

$$G(q,p,\theta,\theta_{\max}) = \ell n(\{(q-p)^2 + qp(\theta_{\max}^2 - \theta^2) + [(q-p)^4 + 2qp(q-p)^2(\theta^2 + \theta_{\max}^2) + q^2p^2(\theta_{\max}^2 - \theta^2)^2]^{\frac{1}{2}}\}/[2(q-p)^2]) \tag{12}$$

Both θ and $\theta_{\max}$ must be small for Eq. (11) to hold true. Similarly we must have $\gamma \le 1 - 12\beta$ so that there is only a negligible chance of intervening plume particles blocking the pairwise radiative interaction specified in Eq. (2). The expansion of $\tau(s)$ in powers of β [see Eq. (5)] makes sense only when β is small but non-negligible. Thus, as a general rule, γ will be nonpositive ($\gamma \le 0.8$, $s_{\min} \ge 0.2$).

Results

Function F has been evaluated using standard numerical techniques, and the results are shown in Figs. 2 and 3. Figure 2 is a graph of F vs $\log(s)$ at $\theta = 0$ for three different values of $\theta_{\max}$. The linear dependence of F on $\log(s)$ is so striking that it demands an explanation. We suppose that the particles only react strongly with their nearest neighbors, so that Eq. (11) can be approximated as

$$F(s,\theta,\theta_{\max},\gamma) \cong \int_{1-\gamma}^{s} \frac{\mathrm{d}p(\gamma + p)^{\frac{4}{3}}}{p} \int_0^{fp} \mathrm{d}\xi \frac{1}{p-\xi}\left(\frac{1}{\gamma + p - \xi}\right)^{\frac{4}{3}} G(p-\xi,p,\theta,\theta_{\max}) + \int_{1-\gamma}^{s} \frac{\mathrm{d}p(\gamma + p)^{\frac{4}{3}}}{p} \int_0^{fp} \mathrm{d}\zeta \frac{1}{p+\zeta} \times \left(\frac{1}{\gamma + p + \zeta}\right)^{\frac{4}{3}} G(p+\zeta,p,\theta,\theta_{\max})$$

$$0 < f \ll 1,\ \xi = p - q,\ \zeta = q - p \tag{13}$$

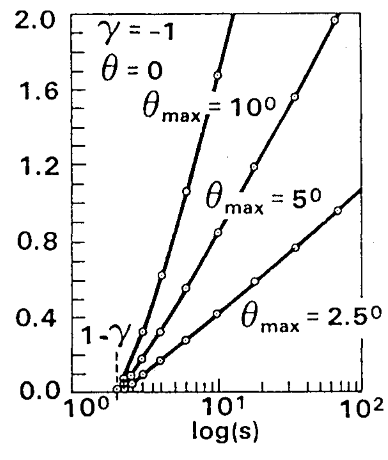

Fig. 2 F down the central axis for different values of $\theta_{\max}$.

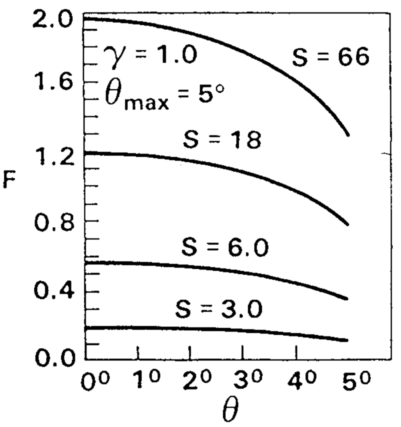

Fig. 3 F vs θ at increasing distance from the rocket motor.

The double integrals will always have $\xi,\zeta \leq fp \ll p$, so Eq. (13) simplifies to

$$F \cong [\ell n s - \ell n(1-\gamma)]M(f,\theta,\theta_{max}) \tag{14}$$

where

$$M = 4f[1 - \ell n(f/\theta_{max})], \quad \text{for } \theta = 0$$
$$= 2f[1 - \ell n(f/\theta_{max})], \quad \text{for } \theta = \theta_{max}$$

Equation (14) justifies the linear dependence of F on log(s) shown in Fig. 2.

Figure 3 is a graph of F against θ for increasing values of s; as s increases, the difference between F at $\theta = 0$ and F at $\theta = \theta_{max}$ also increases. This behavior matches our physical intuition of what is going on inside the plume. Function F is a measure of the difference between a particle's actual temperature and what its temperature would be if it had traveled the same distance without absorbing any of its neighbors' radiation. Obviously, particles traveling down the central axis of the plume at $\theta = 0$ will absorb more radiation—and thus cool off more slowly—than will particles traveling at the plume edge where $\theta = \theta_{max}$. Thus, F will increase with s, and increase more rapidly with s the smaller the value of θ.

Conclusion

Undoubtedly the rocket plume model presented here is in many respects unrealistic; in real plumes the hot particles are not all the same size and do not all travel at the same velocity, and the particle density never drops abruptly to zero at $\theta = \theta_{max}$, as shown in Fig. 1. Moreover, the geometric optics approximation used in Eq. (2) is not always appropriate, especially when the particles are so cool that most of their radiative energy occurs at wavelengths the same size as—or larger than—the particle radius. Nevertheless, the model does reveal two important aspects of the radiant energy exchange between hot particles in a vacuum. First, the size of the parameter $\beta = \varepsilon k R^2 \Psi/12$ indicates when the interparticle radiant energy exchange is important and when it is not. Second, inside the rocket plume, only neighboring particles will exchange significant amounts of radiant energy, and this leads to a linear dependence of τ_B and F on log(s). Both the β parameter and the nearest neighbor energy exchange will have their analogs in any set of equations used to describe the behavior of hot particles traveling in a vacuum. The simplified model developed above should provide useful guidelines for the construction of those more complex plume models required for a realistic simulation of a solid fuel rocket.

Acknowledgment

This work was sponsored by the Department of the Air Force.

References

[1]Pearce, B. E., "An Approximate Distribution of Particle Mass Flux in a High Altitude Solid Propellant Rocket Plume," *AIAA Journal*, Vol. 12, May 1974, pp. 718–719.

[2]Rothwell, W. S. and Schick, H. L., "Emission Spectra from a Stream of Cooling Particles," *Journal of Spacecraft and Rockets*, Vol. 11, Aug. 1974, pp. 597–598.

[3]Worster, B. W., "Particulate Infrared Radiation in Aluminized Solid-Fuel Rocket Plumes," *Journal of Spacecraft and Rockets*, Vol. 11, April 1974, pp. 260–262.

[4]Fontenot, J. E., "Thermal Radiation from Solid Rocket Plumes at High Altitude," *AIAA Journal*, Vol. 3, May 1965, pp. 970–972.

[5]Weast, R. C., ed., *Handbook of Chemistry and Physics*, 51st ed., The Chemical Rubber Co., Cleveland, OH, 1971, pp. A-160–A-218.

Multiburst Cloud Rise: Theory and Experiment

Alan W. Zimmerman*
TRW Defense Systems Group,
Redondo Beach, California

Nomenclature

a = cloud aspect ratio ($2b/\Delta h$)
b = cloud horizontal radius
D = diameter
H = stabilization altitude of cloud
Δh = cloud thickness
K = diffusivity
N = number of bursts
s = burst separation
S = s/D_{FB}
t = time
W = nuclear weapon yield
x = altitude
α = entrainment constant
ν = kinematic viscosity

Subscripts

ES = extended source
FB = fireball
MB = multiburst
O = initial value
PS = point source
SB = single burst

Introduction

THE dimensions and stabilization altitude of the cloud produced by a near-surface nuclear burst are of interest to the weapons systems designer. The gross motion of the cloud, except at very early times, is like that of a buoyant thermal, and solutions[1–4] are available for a single burst. Morton et al.[1] obtained a closed-form solution for a uniformly and stably stratified fluid. This solution was extended in Ref. 5 for the limiting case of many bursts that are closely spaced and nearly simultaneous. It is the purpose of this paper to compare the predictions of this multiburst model with experimental data and with results from hydrocode calculations.

Very limited experimental data are available for multiburst cloud rise. A few two-dimensional hydrocode calculations are also available where the initial source is smeared out in rings or a sheet. These are treated here as "data." The predictions of single cloud rise by inviscid hydrocodes have been found to agree very well with experimental data, even though the dominant mechanism controlling cloud rise and growth is turbulent mixing with its surroundings. A recent survey[6] for the Defense Nuclear Agency of all experiments, calculations, and multiburst models identified those experiments and calculations utilized here for comparison with the model of Ref. 5.

Received April 13, 1987; revision received Sept. 15, 1987.

*Section Head, Member AIAA